Four Quantum Conservation Laws on Black Hole Equilibrium Radiation Process and Quantum Black Hole Entropy

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February 7, 2008

The classical first law of thermodynamic for Kerr-Newmann black hole (KNBH) is generalized to that in quantum form on event horizon. Then four quantum conservation laws on the KNBH equilibrium radiation process are derived. As a by-product, Bekenstein-Hawking's relation $\mathcal{S} = \mathcal{A}/4$ is exactly established. It can be argued that the classical entropy of black hole arise from the quantum entropy of field quanta

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or quasi-particles inside the hole.

PACS number(s): 04.60.+n, 04.20.Me, 97.60.L

It has been nearly one quarter century since Benkenstein[1] and Hawking[2] first showed that the entropy of a black hole is one quarter of its surface area. Despite considerable effort[3] about the quantum[4], dynamic[5], or statistical[6] origin of black hole thermodynamics, however, the exact source and mechanism of the Benkenstein-Hawking black hole entropy remain unclear[7].

By using the brick wall model, 't Hooft[8] identified the black hole entropy with the entropy of a thermal gas of quantum field excitations outside the event horizon. Frolov and Novikov[4] argued that the black hole entropy can be obtained by identifying the dynamical degrees of freedom of a black hole with the states of all fields which are located inside the black hole. Such two opinions differ from each other. A black hole acts as a classical object, but its true microscopic structure still remains unknown.

In this paper, we first derive the thermal spectrum and entropy of a massive complex scalar field on Kerr-Newmann black hole (KNBH) background. From this quantum entropy, we propose a quantum first law of black hole thermodynamics. Then we consider a system in which a KNBH is in equilibrium with this scalar field. We could regard this system as a two-phase equilibrium one which is consisted of radiant phase and black hole phase, with its interface being the event horizon. Further the black hole phase can be thought as a condensed state which

is made up of field quanta inside the hole, while the radiant phase being a excited state of the same field quanta outside the hole. In this thermodynamical picture, the event horizon acts as a membrane[9]. Using thermodynamical method, we obtain four conservation laws on black hole equilibrium radiation process for energy, charge, angular momentum and entropy respectively. The total quantities of these quantum numbers of the whole system are conserved in this stationary thermal equilibrium radiation process. By identifying this complex scalar field with quasi-particles excited by the hole, we propose that the classical entropy of a black hole originate microscopically from the entropy of quanta which constitute the hole.

The general stationary axial solution to Einstein equation is a rotating charged black hole (KNBH) described by three parameters : mass M, charge Q, specific angular momentum $a = J/M$. So we deal with a source-less charged massive scalar field with mass μ and charge q on this background in the non-extreme case ($0 < \varepsilon = \sqrt{M^2 - a^2 - Q^2} \leq M$). (In Planck units system $G = \hbar = c = k_B = 1$).

In the Boyer-Lindquist coordinates, a complex scalar wave function Ψ has a solution of variables separable form[10]:

$$\Psi(t, r, \theta, \varphi) = R(r)S(\theta)e^{(im\varphi - \omega t)} \tag{1}$$

here the angular wave function $S(\theta)$ is an ordinary spheroidal with spin-weight $s = 0$ which satisfies Legendre wave equation[11]:

$$\frac{1}{\sin\theta}\partial_\theta[\sin\theta\partial_\theta S(\theta)] + [\lambda - \frac{m^2}{\sin^2\theta} - (ka)^2\sin^2\theta]S(\theta) = 0, \tag{2}$$

while the radial wave function $R(r)$ is a modified generalized spheroidal wave func-

tion with an imaginary spin-weight, which satisfies the following "modified" generalized spin-weight spheroidal wave equation of imaginary number order[12,13]:

$$\partial_r[(r-r_+)(r-r_-)\partial_r R(r)] + [k^2(r-r_+)(r-r_-) + 2(A\omega - M\mu^2)(r-M) + \frac{[A(r-M)+\varepsilon B]^2}{(r-r_+)(r-r_-)} - (2\omega^2-\mu^2)(2M^2-Q^2) - 2qQM - \lambda]R(r) = 0, \quad (3)$$

where λ is a separation constant, $r_\pm = M \pm \varepsilon, k^2 = \omega^2 - \mu^2, A = 2M\omega - qQ, \varepsilon B = \omega(2M^2-Q^2) - qQM - ma$.

When considering thermal radiation of the KNBH, we need the asymptotic solutions of the radial function $R(r)$ at its event horizon $r = r_+$. In fact, the radial equation has two solutions whose indices at its regular singularities $r = r_\pm$ are $\pm iW$, where W is given blow. The two asymptotic solutions are:

$$R(r) \sim (r-r_+)^{\pm iW}, \quad \text{when} \quad r \to r_+ \quad (4)$$

According to the analytical continued method suggested by Damour-Ruffini[14], these two solutions differ by a extra factor $e^{2\pi W}$. Then, it is easy to obtain a thermal radiation spectrum[15]:

$$< N >= \frac{1}{e^{4\pi W}-1}, \quad W = \frac{\omega - m\Omega - q\Phi}{2\kappa}. \quad (5)$$

where surface gravity $\kappa = (r_+ - M)/\mathcal{A}$, angular velocity $\Omega = a/\mathcal{A}$, electrical potential $\Phi = Qr_+/\mathcal{A}$, reduced horizon area $\mathcal{A} = r_+^2 + a^2$.

In fact, quantum number W is the entropy of quantized scalar fields on KNBH background whose energy satisfies relation:

$$\omega = 2\kappa W + m\Omega + q\Phi = 2\kappa W + \omega_+. \quad (6)$$

This relation demonstrates that a KNBH can have two radiation mechanisms: superradiant mode ($\omega < \omega_+$) and Hawking mode ($\omega > \omega_+$)[16]. In thermal equilibrium radiation process, the hole's surface gravity, angular velocity and electrical potential remain unchanged. By differentiating the energy relation of Eq.(6), we have the following first law of quantum thermodynamics in differential form:

$$\Delta\omega = 2\kappa\Delta W + \Omega\Delta m + \Phi\Delta q. \tag{7}$$

Let us consider a KNBH in thermal equilibrium with a complex scalar field at temperature $\mathcal{T} = \kappa/2$. This system can be thought as a two-phase equilibrium system which is consisted of black hole phase and radiant phase. The black hole phase is a liquid state condensed from ground state field quanta inside the hole, while the radiant phase being a gas phase constituted of excited state field quanta outside the hole. These two phases, as is being interfaced by the event horizon, are two different kinds of states of the same field quanta. In this thermodynamical system, there exists detailed balance process[17], that is, the number of quanta emitted by the hole is equal to that absorbed by it. From conditions of thermodynamical equilibrium on event horizon :

$$\kappa_{r>r_+} = \kappa_{r<r_+}, \Omega_{r>r_+} = \Omega_{r<r_+}, \Phi_{r>r_+} = \Phi_{r<r_+},$$

combining Eq.(7) with the following classical first law of black hole thermodynamics in differential form[18]:

$$\Delta M = \frac{\kappa}{2}\Delta\mathcal{A} + \Omega\Delta J + \Phi\Delta Q, \tag{8}$$

we can deduce four quantum conservation laws for energy, angular momentum, charge and entropy respectively.

$$\begin{aligned} \text{Energy :} \qquad & \Delta M = \Delta\omega, & (9) \\ \text{Angular Momentum :} \qquad & \Delta J = \Delta m, & (10) \\ \text{Charge :} \qquad & \Delta Q = \Delta q, & (11) \\ \text{Entropy :} \qquad & \frac{1}{4}\Delta\mathcal{A} = \Delta W. & (12) \end{aligned}$$

When considering all modes of field quanta, the above conservation relations (9-12) must include sum respect to all possible modes of field configurations. Eqs.(9-12) indicate that a KNBH has discrete increment of energy, angular momentum, charge and entropy. That is, when a black hole emits particles, its energy, charge, angular momentum and entropy are carried away by these quanta , and vice versa. In a stationary thermal equilibrium radiation process, it is reasonable physically conceived that what the hole gains meets with the ends of that the radiation loses. Thus, the total quantities of energy, charge, angular momentum and entropy remain unchanged in this thermodynamic process.

Further, combining Eq.(6) with integral Smarr formulae[18]:

$$M = \kappa\mathcal{A} + 2J\Omega + Q\Phi, \tag{13}$$

we can obtain a special quantum state $m = J, \omega = M/2, q = Q/2, W = \mathcal{A}/4$. This demonstrates that a quantum KNBH is a collection of all possible quasi -particles inside the hole. As quantum number m, ω, q, W are discrete numbers, not only the parameters $J, M, Q, \mathcal{A}$ but also $\Delta J, \Delta M, \Delta Q, \Delta\mathcal{A}$ take discrete values. So, a

KNBH could be thought as a condensed phase consisted of all possible modes of bosonic field quanta.

In fact, Eq.(12) is a generalized second thermodynamic law in quantum form. By integrating this equation, we obtain quantum black hole entropy:

$$W = \frac{1}{4}\mathcal{A} + C. \tag{14}$$

As Bekenstein-Hawking's classical black hole entropy[1,2] : $\mathrm{S} = \mathrm{A}/4 = \pi\mathcal{A}$, the quantum entropy W is equal to the reduced entropy $W = \mathcal{S} = \mathrm{S}/(4\pi)$, so we have Bekenstein-Hawking relations (Choose constant ($C = 0$):

$$\mathcal{S} = W = \frac{1}{4}\mathcal{A}. \tag{15}$$

Eq.(15) shows that Bekenstein-Hawking black hole entropy is equal to quantum entropy of a complex scalar field. In other words, the classical entropy of black holes originates statistically from quantum entropy of quantized fields.

The point of our view that a black hole is being made up of radiation field quanta can be simply illustrated by a soap bubble thermodynamical model. Let us consider a sphere with radius $r = 2M$, mass M in vacuum, this sphere contains uniform radiation inside it. A condition that this bubble doesn't break down is that its pressure p must be equal to its surface tension σ being divided by its mass M, that is, $p = \sigma/M = \rho/3$, where $\rho = 3/(32\pi M^2)$ being the mean density of radiations in this sphere. So we have a relation $\sigma = 1/(32\pi M)$. Then the surface gravity $\kappa = 8\pi\sigma = 1/(4M)$, which is exactly equal to the surface gravity in a Schwarzschild black hole case. Thus, the radiant pressure can contend with

the gravity of a sphere symmetric black hole. (Note: The factor 8π rises from Einstein coupled coefficient, and $\rho = 3p$ is state equation of photon radiations in our argument). On the other hand, the energy of radiations inside the sphere can't take arbitrary values, it must be multiple of ground state energy $8\pi M$ which is due to Heisenberg uncertain principle.

In summary, we regard a Kerr-Newmann black hole as a classical two-phase thermodynamical system. Using two-phase thermodynamical equilibrium condition, we derive four quantum conservation laws on black hole equilibrium radiation process. The total energy, total charge, total angular momentum, total entropy of the whole system are conserved in this process. By identifying a KNBH with a collection of condensed ground state quanta, we propose that the classical entropy of a black hole originate microscopically from quantum entropy of ground state quanta inside the hole.

This work is supported partly by the NNSF and Hubei-NSF in China.

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